Annotated Examples: M203 Polynomial and Rational Expressions, Functions, and Equations

The examples that follow are intended to illustrate how the learning principles are used to support students' engagement with the content and practices outlined in this badge. These examples do not provide comprehensive coverage of those expectations, but rather elevate some of the learning principles that are less likely to be part of published curricular materials for mathematics instruction. The examples that follow were developed by the Math Badging writing team, unless otherwise specified. These are a small sample of types of learning activities that can be done with students, both in and out of a traditional classroom setting.

Example 1: Solar System

Students are given a task like this:

Part 1: Make Sense

In the early 1600s, Johannes Kepler (1571–1630) studied the motions of the planets to find a good mathematical model for them. In 1619, he published his third law of planetary motion, which says how the orbital periods of the planets are related to their distances from the sun. In Kepler's time, Uranus and Neptune had not been discovered, but in the table below are data for all 8 planets:

planet	distance (millions of km)	period (days)
Mercury	57.9	88.0
Venus	108.2	224.7
Earth	149.6	365.2
Mars	227.9	687.0
Jupiter	778.6	4,331
Saturn	1,433.5	10,747
Uranus	2,872.5	30,589
Neptune	4,495.1	59,800

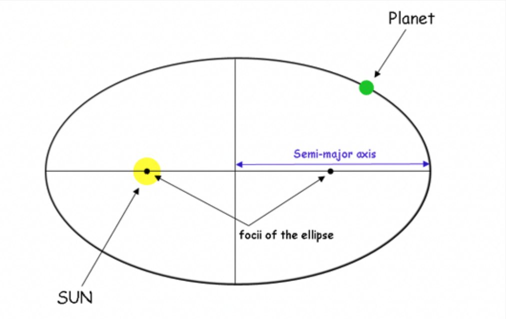
1. *What sort of function do you anticipate will fit this data? How did you arrive at that conclusion?*

Part 2: Graphing

2. *Plot the distance (x) and period (y) of each planet, and find a polynomial model that best fits the data. You may have to experiment with both the degree of your polynomial function and the number of terms.*
3. *Let's try a different approach. Make another plot that uses the square root of each distance. Find a polynomial model that best fits the data.*
4. *Which model do you think is best?*
    a. *Jupiter has a lot of moons. Look up the periods and distances of some of them, and use that data to make a polynomial model of the relationship between period and distance for Jupiter's moons.*
    b. *Use your model to predict the period of another one of Jupiter's moons using the radius of its orbit.*
    c. *How good was the prediction? What are some possible sources of error?*

Part 3: Analyze

Kepler's Third Law of Planetary Motion states: "The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit."



[Keplers laws]

5. *What does Kepler's Third Law look like in terms of an algebraic equation? What do variables in that equation mean?*
6. *How can you use the information you gathered from your research to determine the value of the constant?*
7. *Does the same constant hold for every planet?*
8. *How might this information be helpful in determining what polynomial function models this relationship?*

Part 4: Refine and Reflect

Jupiter has a lot of moons. The Galilean moons were discovered in 1610; they are IO, Europa, Ganymede, and Callisto.

9. *Another moon of Jupiter, number LXXI, was discovered in 2018. Its distance from Jupiter is 11,483 thousand km. According to your model, what should its period be?*
10. *The actual period of the moon (LXXI) is 252.0 days. How close was the prediction? What are some possible sources of error?*
11. *What is something you learned about the work of developing functions that model real-world phenomena?*

Sample Learning Activity

Part 1: Make Sense

Launch this experience by inviting students to consider why studying the motions of the planets might be important. Consider using these questions:

- *What might the motion of the planets tell us?*
- *What factors do you think influence the motion of the planets?*

To deepen understanding on why studying planet motion is important, consider having students explore one or more of these resources:
Video: How the Movement of Other Planets Affects Earth — Yes, Really
Article: Planet Movements Impact Earth

Ask students:

- *What is something you learned that was new?*
- *What surprised you?*

After discussing, present students the prompt and table above. Invite students to make sense of the table given:

- *What do you think the term "period" means here?*
- *What do you notice about the distance and period? Is there any relationship between the two?*
- *What type of association is represented in the table (linear, quadratic, power, etc.)?*

After discussing, let students know that we are working with this information to find a polynomial function that best fits the data. Give students time to consider what sort of function might fit this data and why, then ask them to share thoughts with their classmates, in pairs or small groups. Make note of students' ideas in a public space to refer to as the opportunity surfaces.

Part 2: Graphing

Ensure students have access to graphing technology, or share a link to the Desmos graph Planetary Distance and Period. As students work on this part, monitor and select students to share their approaches with the class.

Questions that can be posed to student pairs as they work:

- *What kind of growth is happening in the table? How do you know?*
- *What happened to your best fit equation as the degree changed?*
- *How would you explain the relationship between the period and distance of the moons?*

Consider using questions like this to facilitate discussion:

- *What's promising about this approach?*
- *What questions does this approach surface for you?*
- *What can you take from this approach to apply to your own work?*

Provide time for students to revise and refine their work.

Part 3: Analyze

Provide time for students to look up the information about Jupiter's moons. You may provide links for students such as NASA Jupiter Moon, MSU.edu Jupiters moons, or another site.

Invite students to share their models for the relationship between the orbital period and distance, and for the relationship between the orbital period and the square root of the distance.

Share with students the statement of Kepler's Third Law, which is: "The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit." This means that the square of the orbital period is the product of some constant, k, and the cube of the semi-major axis (distance from the center of Jupiter). This can be expressed as $p^2=k \cdot d^3$. This can be rearranged into something of the form $p= k(d^{1/2})^3$ which is the relationship they have been investigating. Once Kepler saw this relationship, he could divide the square of any planet's period by the cube of its distance in order to find out the proportionality constant.

The goal of this part of the journey is for students to explore these ideas for themselves and verify that all planets have the same proportionality constant. You may also invite students to compare what happens with Jupiter's moons—do they have the same proportionality constant as the planets? Newton's laws of gravity later explained why this relationship between distance and period is true. Give students time to explore these ideas by conducting additional research.

Monitor and select students to share their approach as they work on these prompts. After some quiet time to think, encourage students to work with a partner or in small groups. Consider using the following questions to facilitate the discussion:

- *What's promising about this approach?*
- *What questions does this approach surface for you?*
- *What can you take from this approach to apply to your own work?*

Provide time for students to revise and refine their work.

Part 4: Refine and Reflect

Give selected pairs of students time to share responses and invite classmates to respond or pose questions.

Provide time for students to revise their work. While they work, select additional pairs to share how they used the structure and relationship between the equations to determine which sets of equations have the same answer. Give students additional time to revise their work.

In this example, students are:

- given a cognitively demanding task. Students are asked to make conjectures about the relationship between a planet's distance from the sun and period. Students have multiple entry points to reason about this prompt in various ways (LP 1).

- using the varied structures that give opportunities to determine an approach by working independently, then sharing with a partner or class, and revising their response based on those interactions (LP 2).
- making sense of another person's line of mathematical reasoning as well as critiquing the mathematical reasoning that surfaces. Students build conceptual understanding as they work on their knowledge of polynomials, equations, and functions (LP 3).
- given the opportunity to develop their mathematical identity since their ideas and reasoning are the foundation of the discussion (LP 4).
- developing a sense of identity and agency as they work to make sense of planetary motion and Kepler's Third Law, gaining insight into the work various career paths do when using mathematics to model relationships (LP 5 and LP 6).
- leveraging technology to explore and determine the polynomial function to best fit the data given (LP 7).

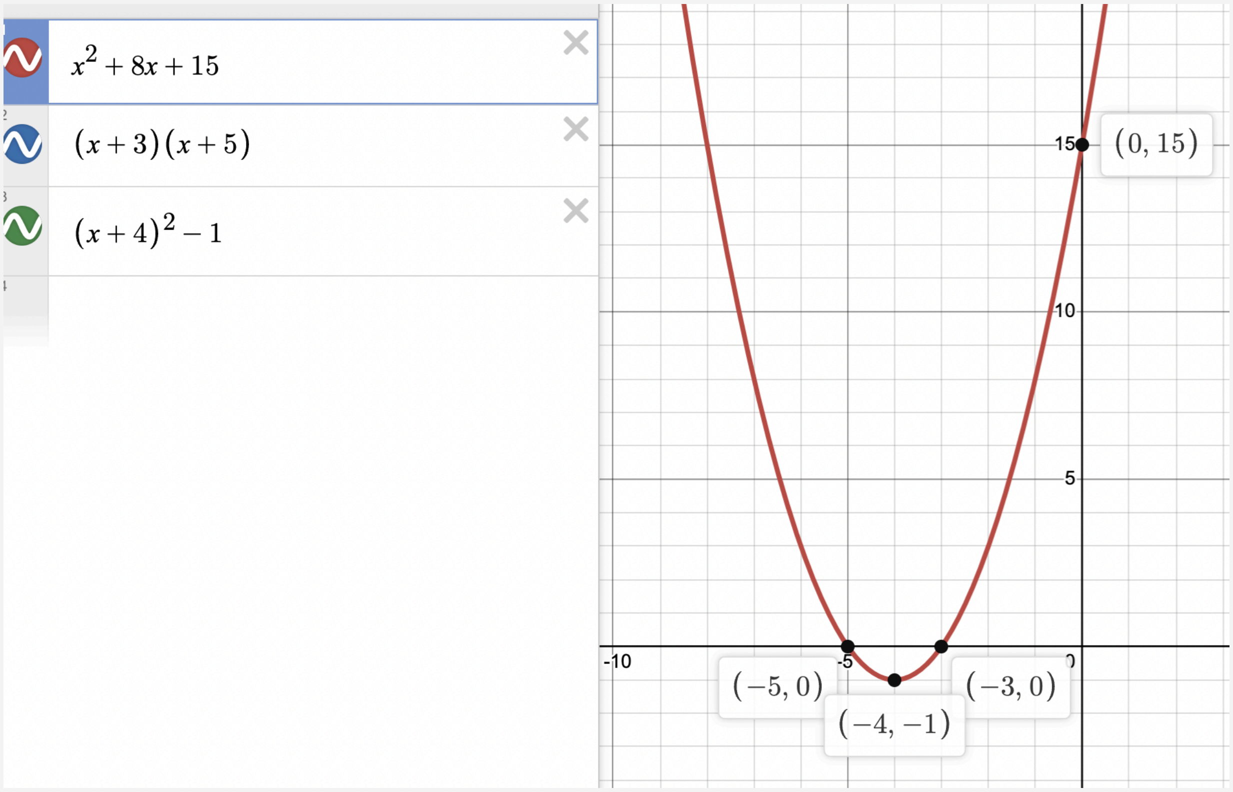
Through the implementation of this learning activity, look for opportunities to support students' growth and development on this XQ competency:

- **Scientific Investigation:** Modeling systems (FL.MST.4.c)

Example 2: Equivalent But Not the Same

Students are given a task like this:

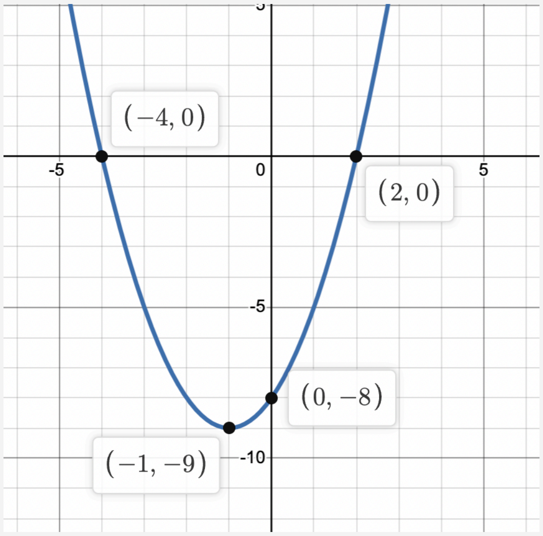
You have been given a polynomial graph with three equivalent equations. Analyze and make connections from the information in the questions below.



1. *Complete the chart below using the given graph and equations*

Write in each equation according to its form.	*Explain the features of the equation connecting the graph to this form.*	*When would using this form of the equation be most useful?*
Standard Form		
Vertex Form		
Factored		

2. *Now, you have been given the graph of a polynomial function, but the equations are missing.*



a. *Write an equation for the factored form. What do the values for x represent in this equation? Defend your thinking using mathematical and/or written justifications.*

b. *Explain the process for developing the vertex form of the function. What does the value for x represent in this equation?*

c. *Sketch the graph for your own polynomial equation using Desmos, linked here.*
d. *From the graph, write the three equivalent equations that would produce this graph.*
e. *Defend your thinking using mathematical and/or written justifications.*
f. *Summarize the importance of three forms of a polynomial function.*

Sample Learning Activity

Provide time for students to consider the question posed and independently develop a response. Have students write their response including their justification. Have partner pairs share their written justification and critique each other's response, allowing for students to defend their response. Provide time for students to revise their written justifications using the information gained during their partner-sharing time. While observing student discussions, identify varying solution methods to be presented during the whole group sharing. Have selected students present their responses—consider using a document camera for students to show actual work—and explain their justification.

As discussion takes place, invite classmates to join the discussion. Ask:

- *What about this reasoning makes sense?*
- *Is there anything you disagree with?*
- *What questions surface for you?*

Provide time for students to revise their work.

In this example, students are:

- given a cognitively demanding task. They are asked to analyze polynomial functions given points on the graph and corresponding numbers in the equation. Students have multiple entry points to reason about these equations in various ways (LP 1).
- working collaboratively in pairs to complete the task, allowing for social interaction with peers. Students are also encouraged to ask questions to their peers to understand each other's thinking process and revise their own thinking (LP 2).
- working to make sense of the relationships between the standard, factored, and vertex forms of an equation. In so doing, students learn to recognize that different forms are helpful for identifying key information about the graph (LP 3).
- leveraging technology to examine a polynomial function and to support their conceptual understanding of intercepts, vertexes, and equivalent forms of an expression (LP 7).

Through the implementation of this learning activity, look for opportunities to support students' growth and development on this XQ competency:

- **Logical Thinking:** Recognizing patterns (OT.Crt.2.e)

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